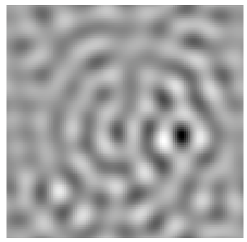
Fourier Analysis

PSY 310

Greg Francis



Lecture 08

It's all waves!

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Representation of information

- One of the *big issues* in perception (and cognition in general) is identifying how the brain represents information
- In today's lecture I want to do two things
  - 1) Discuss a bit about ways of representing information
  - 2) Discuss a mathematical property that is a commonly used tool in perception
- Fourier analysis
- Don't freak out!

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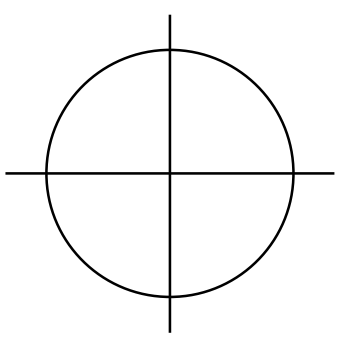
Joseph Fourier

- French mathematician (1768-1830)
- Involved in the French revolution
- An administrator for Napolean Bonaparte
- Discovered a mathematical property that even other mathematicians initially found difficult to accept

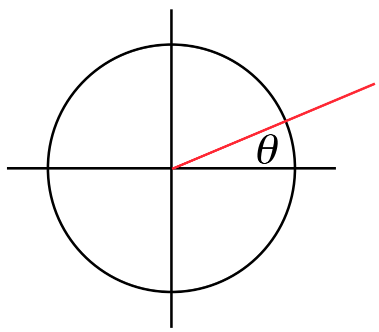
Sines and cosines

- Take a circle with it's center at position (0,0)



Sines and cosines

- Take a circle
- Draw a line from the middle at an angle



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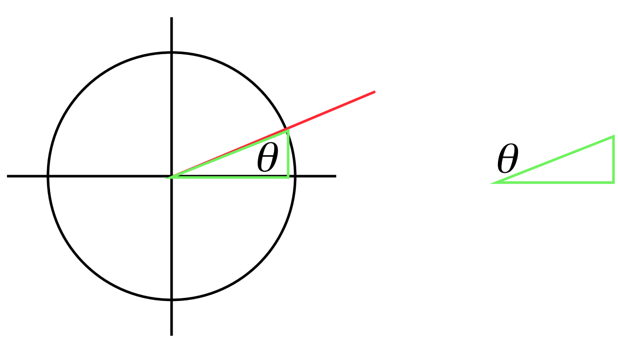
Sines and cosines

- Take a circle
- Draw a line from the middle at an angle
- Draw the triangle where the line crosses the circle



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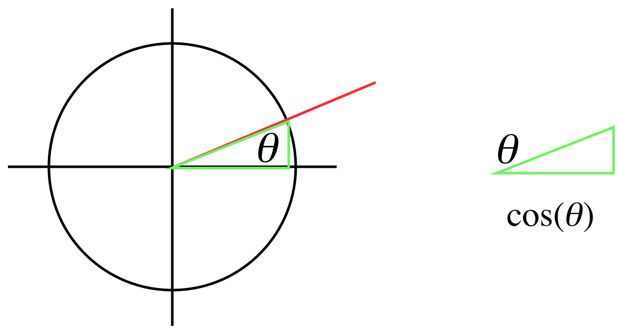
Sines and cosines

- The cosine function cos(θ) is the proportion of the radius of the circle in the x-direction



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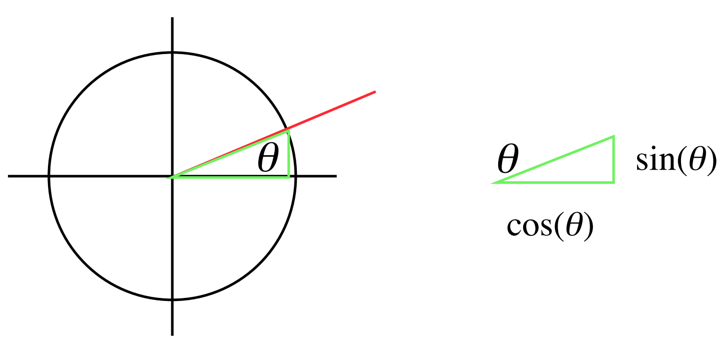
Sines and cosines

- The cosine function cos(θ) is the proportion of the radius of the circle in the x-direction
- The sine function sin(θ) is the proportion of the radius of the circle in the y-direction



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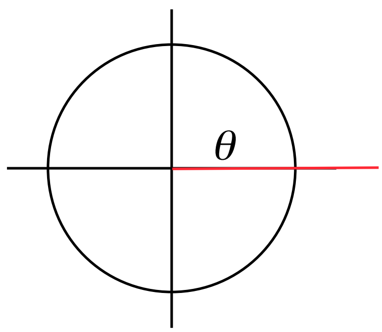
Sines and cosines

- So



$$\cos(0) = 1$$

$$\sin(0) = 0$$

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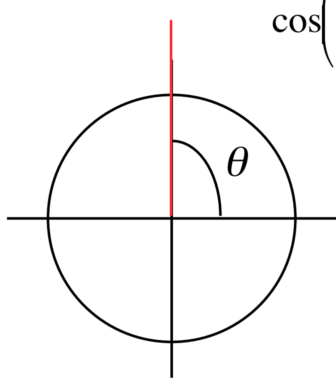
Sines and cosines

- So

$$\cos\left(\frac{\pi}{2}\right) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1$$



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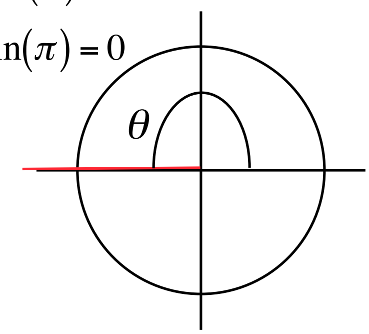
Sines and cosines

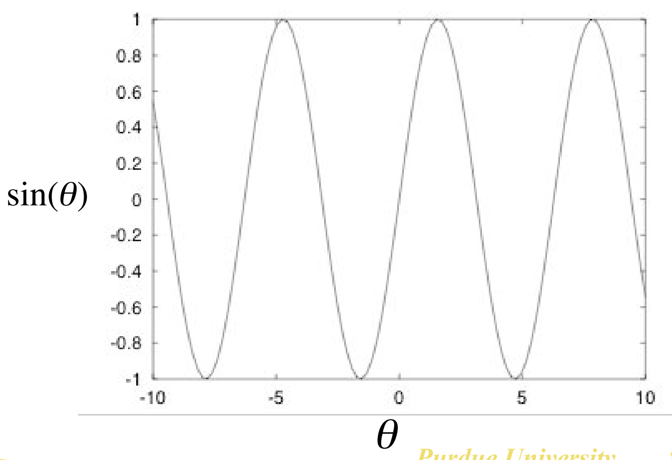
- So

$$\cos(\pi) = -1$$
$$\sin(\pi) = 0$$



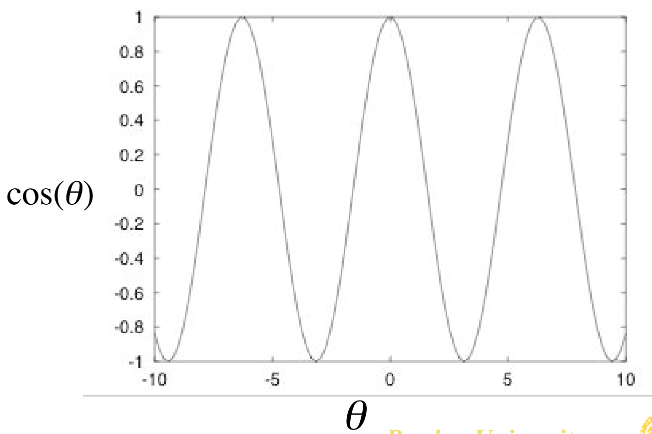
Sine function

- If you plot the sine function as theta varies you get this nice wave



Cos function

- If you plot the cosine function as theta varies you get this other nice wave



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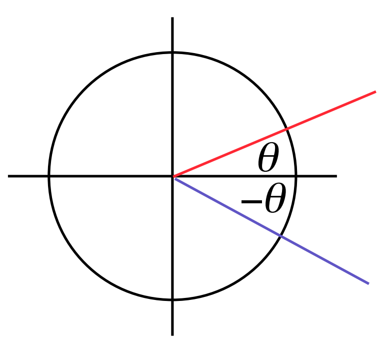
Properties

- There are some interesting properties of these functions

$$\cos(\theta) = \cos(-\theta)$$



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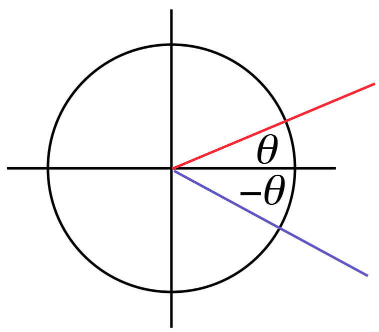
Properties

- There are some interesting properties of these functions

$$\sin(-\theta) = -\sin(\theta)$$



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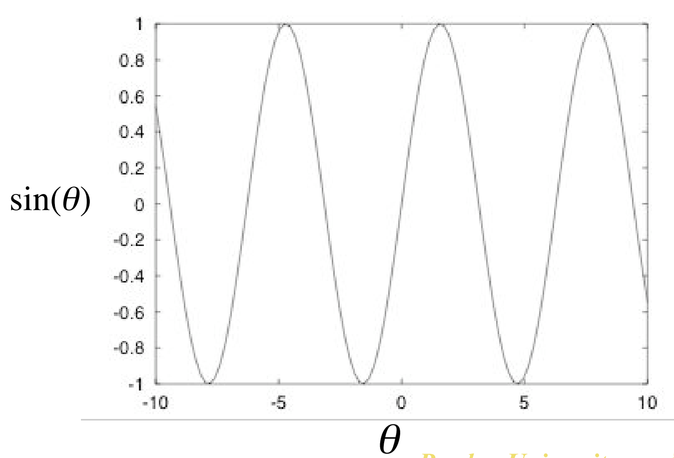
Sine function

- You can speed up the wave by multiplying theta



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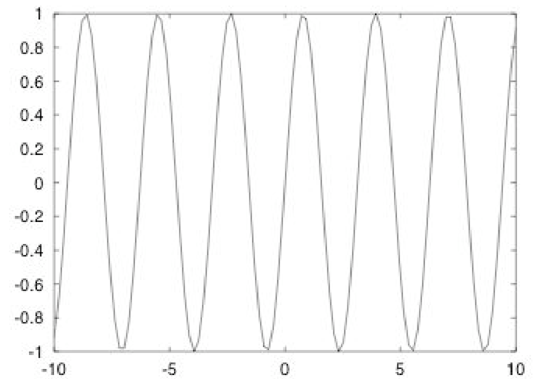
Sine function

- You can speed up the wave by multiplying theta

$\sin(2\theta)$



θ

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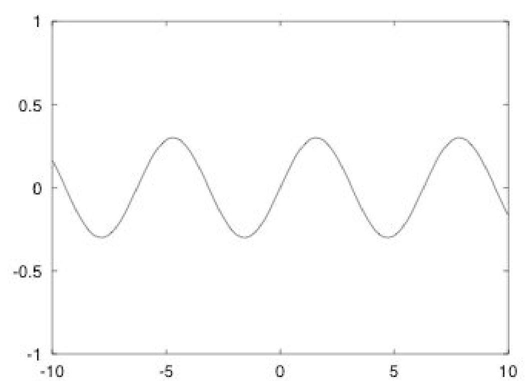
Sine function

- You can also change the height of the wave
- Amplitude

$0.3\sin(\theta)$



θ

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Frequency

- Suppose you are only interested in theta values between -Pi and Pi
  - You can easily generalize to other ranges, but the equations look worse
- In equations of the form

$$\sin(n\theta) \qquad \cos(n\theta)$$

- We say the frequency of the wave is *n*
- This is how many times the wave cycles (comes back to its starting value)

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Orthonormality

- This property is harder to prove, but easy to show examples
- If you multiple any sine and cosine functions and take the integral, you get either zero or pi

$$\int_{-\pi}^{\pi} \sin(n\theta)\cos(m\theta)d\theta = 0$$

$$\int_{-\pi}^{\pi} \cos(n\theta)\cos(m\theta)d\theta = \begin{cases} \pi \text{ if } n = m \\ 0 \text{ if } n \neq m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(n\theta)\sin(m\theta)d\theta = \begin{cases} \pi \text{ if } n = m \\ 0 \text{ if } n \neq m \end{cases}$$

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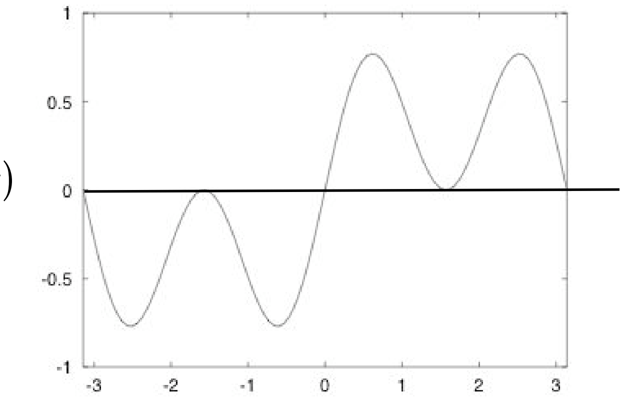
Orthonormality

- Note the positive parts are mirror image of the negative parts

$\sin(2x)\cos(x)$



x

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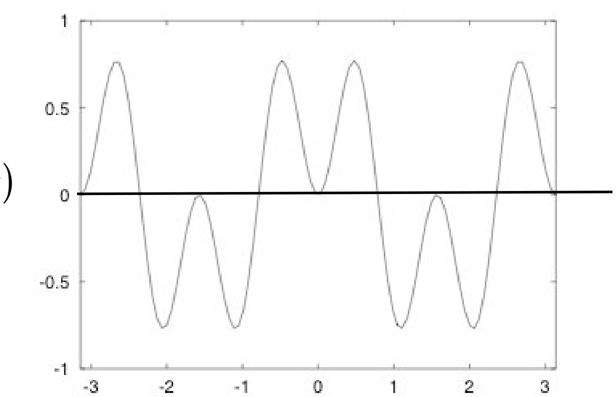
Orthonormality

- Note the positive parts are mirror image of the negative parts

$\sin(2x)\sin(4x)$



x

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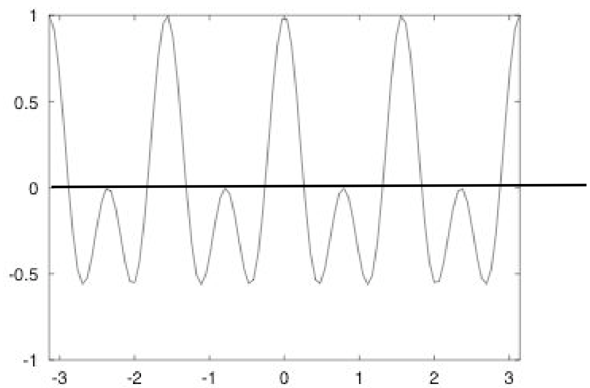
Orthonormality

- It's less obvious here, but each positive part covers two negative parts

$\cos(2x)\cos(6x)$



x

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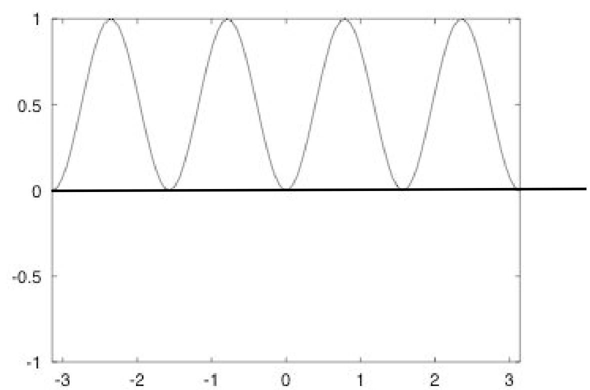
Orthonormality

- If the functions are the same, the integral equals pi

$\sin(2x)\sin(2x)$



x

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Orthonormality

- We can sort of ask how much of one function is made by another function
- E.g., sin(2x) has no part of it that is made of sin(3x), or cos(7x)
- What about other functions? How much of the function is made up of sine or cosine functions of different frequencies?
- Linear
  - $f(x) = mx+b$
- Parabola:
  - $f(x) = ax^2+b$
- Whatever

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Fourier

- Proved that you can write (almost) any function as a series of properly weighted (amplified) sine and cosine functions of different frequencies

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

- The trick is to find the proper values of a_0, a_n, and b_n

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Finding coefficients

- The first term is fairly easy to find, it's just the average

$$a_0 = \frac{1}{2}\int_{-\pi}^{\pi} f(x)dx$$

- To find the other a_n terms, just multiply by a cosine function of the n-th frequency

$$a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

- To find the b_n terms, just multiply by a sine function of the n-th frequency

$$b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

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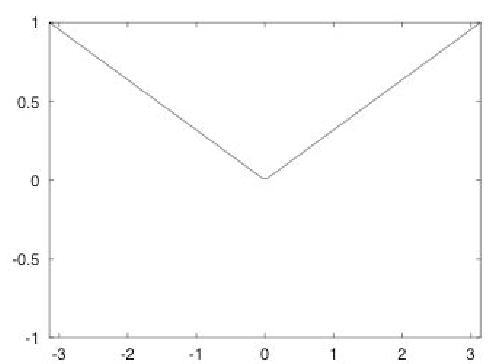
Example

- Suppose our function is the absolute value function
  - Divided by pi to keep everything between -1 and +1.

$$f(x) = \frac{|x|}{\pi}$$



x

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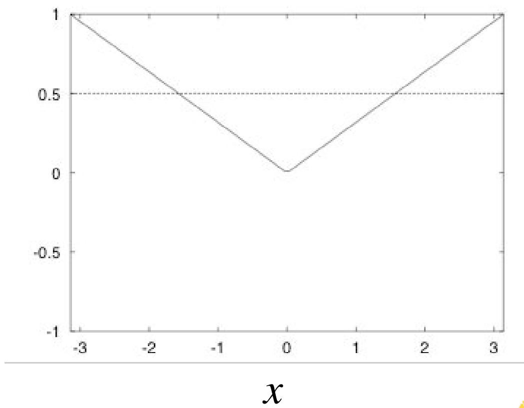
Example

- Suppose our function is the absolute value function
  - Divided by pi to keep everything between -1 and +1.

$$a_0 = \frac{1}{2}$$

$$f(x) = \frac{|x|}{\pi}$$



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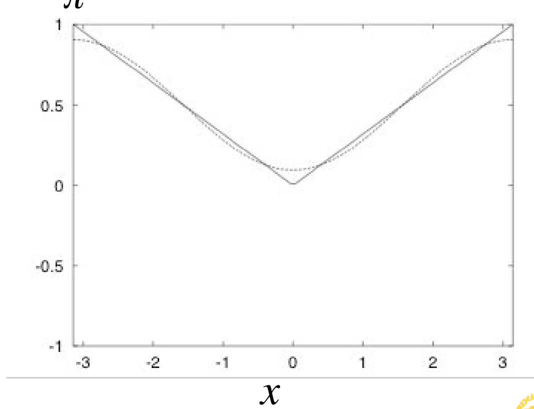
Example

- Suppose our function is the absolute value function
  - Divided by pi to keep everything between -1 and +1.

$$a_0 = \frac{1}{2} \qquad a_1 = \frac{-4}{\pi^2} \qquad a_0 + a_1 \cos(x)$$

$$f(x) = \frac{|x|}{\pi}$$



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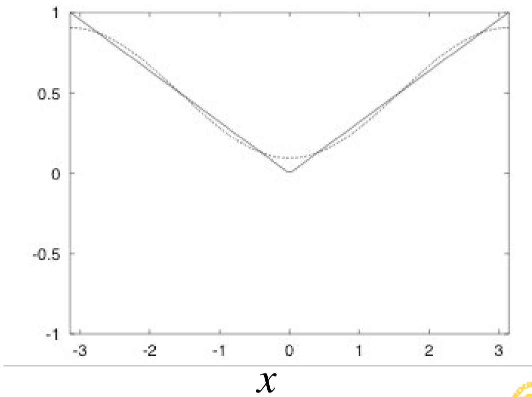
Example

- Suppose our function is the absolute value function
  - Divided by pi to keep everything between -1 and +1.

$a_0 = \frac{1}{2}$ $a_1 = \frac{-4}{\pi^2}$ $a_2 = 0$

$a_0 + a_1 \cos(x)$

$f(x) = \frac{|x|}{\pi}$



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Example

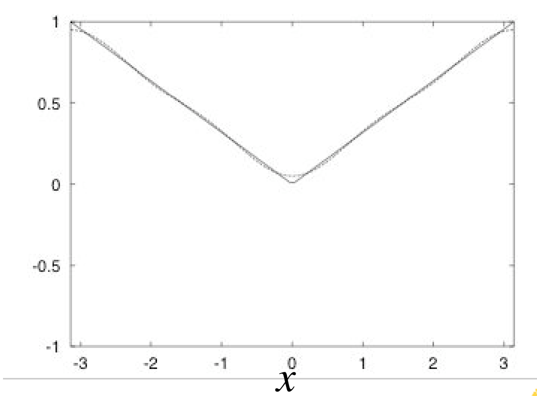
- Suppose our function is the absolute value function
  - Divided by pi to keep everything between -1 and +1.

$a_3 = \frac{-4}{(3\pi)^2}$

$a_0 = \frac{1}{2}$ $a_1 = \frac{-4}{\pi^2}$ $a_2 = 0$

$a_0 + a_1 \cos(x) + a_3 \cos(3x)$

$f(x) = \frac{|x|}{\pi}$



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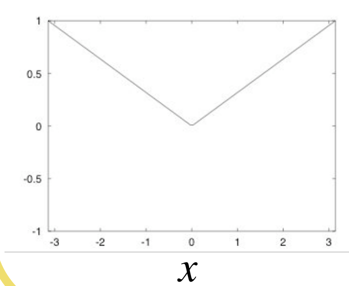
Why does this matter for perception?

- (1) We are curious as to how the brain *represents* information
- (2) We can do essentially the same thing 2-dimensions
- (3) It is useful to analyze images and receptive fields in terms of the spatial frequency

Representation

- A function can be described either in space (x)

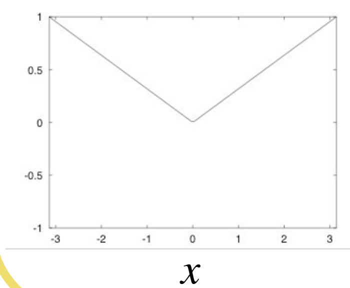
$$f(x) = \frac{|x|}{\pi}$$

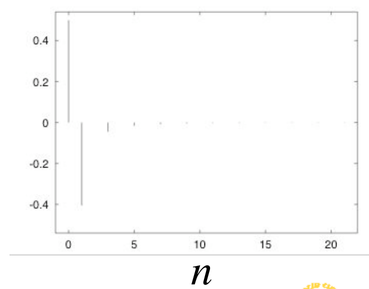


Representation

- A function can be described either in space (x)
- Or in terms of the Fourier coefficients (a_0, a_1,…,b_1, b_2,…)

$$f(x) = \frac{|x|}{\pi} \qquad a_0 = \frac{1}{2} \quad a_n = \frac{-4}{(n\pi)^2} \qquad b_n = 0$$





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Representation

- Perhaps the brain just converts the spatial image into something like Fourier coefficients
- No loss of information!

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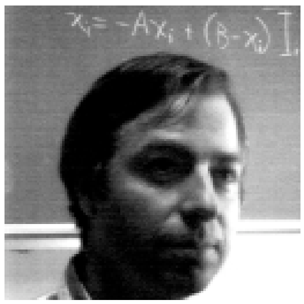
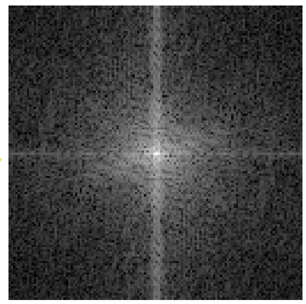
2-dimensional images

- It's a bit more complicated, but you can do essentially the same kind of analysis with 2-D images

Spatial image



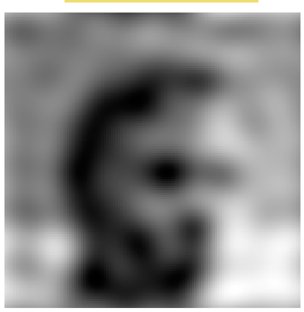
Fourier components



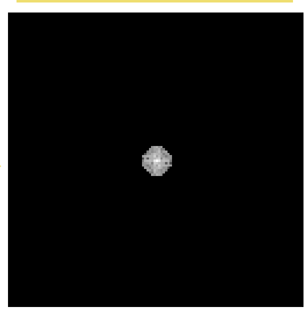
2-dimensional images

- You can isolate the low frequency information in the image

Spatial image



Fourier components

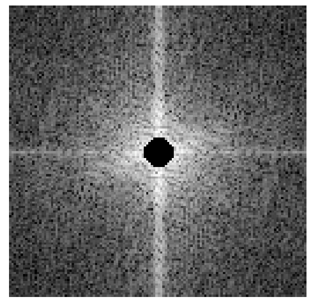


2-dimensional images

- You can isolate the high frequency information in the image

Spatial image

Fourier components



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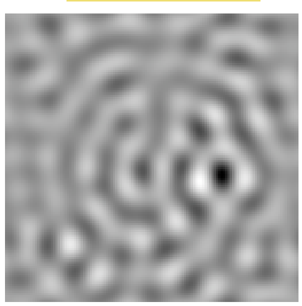
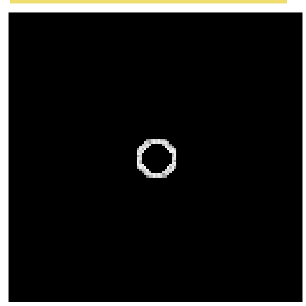
2-dimensional images

- You can isolate any range of frequencies you want

Spatial image



Fourier components



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Conclusions

- Fourier analysis
- You can represent any function (image) as a combination of sine and cosine functions
- In principle, the brain might convert visual images into some completely different representation of information

Next time

- The visual system's response to spatial frequency
- Receptive fields
- Images